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LETTER TO THE EDITOR

A Skyrme-like lump in two Euclidean dimensions

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Abstract. Models consisting of a complex scalar field in $\mathbb{R}_2$ and featuring symmetry-breaking potentials, are proposed. These models possess topologically stable classical lumps. For a special subclass of these models, explicit solutions for these lumps can be found. The possibility of treating these lumps as solitons in $(2+1)$ dimensions is briefly remarked on.

There is currently considerable interest in field theories defined on $(2+1)$-dimensional Minkowski space, on the one hand [1] because of the fundamental insights into quantum field theory that such models offer, and on the other hand [2] because of their practical relevance to problems in condensed matter physics.

In these investigations [1, 2], the model used is the O(3) nonlinear sigma model, which at the classical level is equivalent to the $\mathbb{CP}^1$ model. The distinguishing feature of the O(3) (and the $\mathbb{CP}^1$) model is that it possesses topologically stable finite action (instanton) solutions discovered by Belavin and Polyakov [3], and these topological solutions are the static field configurations of the theory in $(2+1)$ Minkowski space.

From a certain viewpoint, one can regard the $\mathbb{CP}^1$ field theory in $(2+1)$ dimensions as the solution [4] theory of the $\mathbb{CP}^1$ lump moving in the $\mathbb{R}_2$ space. Some classical aspects of this problem are studied by Forgács *et al* [5] as well as by Ward [6] and Stokoe and Zakrzewski [7] using the method of Manton [8]. What makes the $\mathbb{CP}^1$ model unusual as a solitonic [4] model is that, unlike for example the φ^4 and the sine-Gordon models, it is not endowed with a symmetry breaking potential and hence its 'soliton' cannot be localised to an *absolute* scale in $\mathbb{R}_2$. Indeed, due to the scale invariance of the $\mathbb{CP}^1$ system in $\mathbb{R}_2$, the scales of its lumps (namely the instantons) are arbitrary.

This lack of an absolute localisation scale has an important consequence in the quantum theory [9]. Since instantons of arbitrary scales overlap, a dilute-gas approximation involving only unit topological charge instantons cannot be justified; hence it becomes necessary to take into account all instantons. This last circumstance can give rise to problems in the treatment of the quantum $\mathbb{CP}^1$ theory in $(2+1)$ dimensions, as pointed out by Din and Zakrzewski [10].

This brings us to the subject of the present letter. We propose an alternative for the nonlinear sigma models in $\mathbb{R}_2$, which also has topologically stable finite action solutions that are, however, localised to an absolute scale. We envisage that this localised lump may be treated as the soliton of the corresponding field theoretic model in $(2+1)$ dimensions.

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The model, which consists of a complex scalar field $\varphi(x_i, t)$, x_i in $\mathbb{R}_2$, has two salient features. The first is that the Lagrangian includes a symmetry breaking potential, which exhibits a dimensional constant setting the absolute scale of the topologically stable lump solution in $\mathbb{R}_2$, in the static case. The second, and most crucial, feature is that it is endowed with a special quartic kinetic term. This is a Skyrme-like term that overcomes the well known obstacle due to Derrick's virial theorem [4]. (There would be no objection to the presence of a quadratic kinetic term in addition, but this is not necessary.)

In its most general form, the Lagrangian density is

$$\mathscr{L} = -\tfrac{1}{2}(\mathrm{i}\partial_{[\mu}\varphi\partial_{\nu]}\varphi^*)^2 + f^2(\eta, |\varphi|)|\partial_\mu\varphi|^2 - \lambda^2 V(\eta, |\varphi|) \tag{1}$$

where $x_\mu = t$, x_i, with $x_i = (x_1, x_2)$ in $\mathbb{R}_2$, and the metric diag$(1, -1, -1)$ for $(2+1)$-dimensional Minkowski space. λ and η are parameters with the appropriate dimensions, $f(\eta, |\varphi|)$ is some positive function with a suitable asymptotic behaviour in $\mathbb{R}_2$, consistent with *finite energy*†, and $V(\eta, |\varphi|)$ is a symmetry breaking potential†. The square brackets in the quartic kinetic term denote antisymmetrisation, which implies that no power other than the square of the 'velocity' field $\partial_0\varphi$ will appear in (1). As a result, (1) can be looked at as a realistic dynamical model. The function $f(\eta, |\varphi|)$ may be taken to be independent of φ, or even equal to zero.

In the special case where $f = (\eta^2 - |\varphi|^2)^2$ and $V = (\eta^2 - |\varphi|^2)^4$, the static version of (1)

$$\mathscr{L}_0 = -[\tfrac{1}{2}(\mathrm{i}\partial_{[i}\varphi\partial_{j]}\varphi^*)^2 + f^2(\eta, |\varphi|)|\partial_i\varphi|^2 + \lambda^2 V(\eta, |\varphi|)] \equiv -\mathscr{S} \tag{2}$$

coincides with a subsystem of a model on $\mathbb{R}_2$, derived by dimensionally reducing [11] the generalised Yang-Mills system [12]

$$S = \int \mathrm{tr}\, F^2_{MNRS} \tag{3}$$

$$F_{MNRS} = \{F_{MN}, F_{RS}\} + \mathrm{cycl}(NRS) \tag{3$'$}$$

on $\mathbb{R}_2 \times S^6$. It is interesting to note that the corresponding subsystem of the model derived from a similar dimensional reduction of the (usual) Yang-Mills system on $\mathbb{R}_2 \times S^2$ would have yielded the φ^4 model, which has topologically stable (kink) solutions [4] in $\mathbb{R}_1$ but not in $\mathbb{R}_2$.

This and other detailed properties of (1) and (2) will be given elsewhere. In the present letter, we restrict our attention mainly to the classical properties of the solutions of the field equations of (2), and comment briefly on some of the quantum mechanical properties of (1).

(I) *Classical lumps.* Perhaps the most remarkable feature of the solutions of the Euler-Lagrange equations of the static system $\mathscr{L}_0$ given by (2) is that they coincide with the vortex-like field configurations

$$\varphi(x) = \eta R(r)\, \mathrm{e}^{\mathrm{i}n\theta} \tag{4}$$

with $r = \sqrt{x_i x_i}$, $\theta = \tan^{-1}(x_2/x_1)$, despite the fact that there is no U(1) (Maxwell) field in $\mathscr{L}_0$. Clearly, this is a result of the special dynamics of the Skyrme-like term in L_0.

That solutions of the form (4) exist, follows immediately from the fact that the action functional of (2), for the field configurations (4), has a manifestly positive

† See equation (6*a*, *b*) for restrictions on f and V.

definite expression. Such solutions must, of course, satisfy the finite-energy and smoothness conditions

$$R(r) \xrightarrow[r\to\infty]{} 1 \tag{5a}$$

$$R(r) \xrightarrow[r\to 0]{} 0. \tag{5b}$$

The asymptotic condition (5*a*) can be related to the topological stability of the solution (4), by considering the following two inequalities:

$$(\lambda\sqrt{V} - \mathrm{i}\varepsilon_{ij}\partial_i\varphi\partial_j\varphi^*)^2 \geqslant 0 \tag{6a}$$

$$|f\partial_i\varphi - \mathrm{i}f\varepsilon_{ij}\partial_j\varphi^*|^2 \geqslant 0. \tag{6b}$$

Provided that $\sqrt{V}$ and f are polynomials in $|\varphi|$, it is easy to see that the cross-terms in (6*a*, *b*) will be total divergences. As a consequence of the inequalities (6*a*, *b*) then, we find the important inequality

$$\int_{R_2} \mathscr{S} \geqslant \int_{R_2} \partial_i\Omega_i \tag{7}$$

where the total divergence $\partial_i\Omega_i$ is just the sum of the two cross-terms referred to above. For example, with the simplest choice of $V = (\eta^2 - |\varphi|^2)^2$, we have $\Omega_i = -\lambda\mathrm{i}\varepsilon_{ij}(2\eta^2 - |\varphi|^2)\varphi^*\partial_j\varphi$.

The right-hand side of (7) can be written as a line integral, and provided we require the asymptotic condition

$$|\varphi| \xrightarrow[r\to\infty]{} \eta \tag{8}$$

this line integral will yield a winding number, which provides the topological lower bound for the integral on the left-hand side of (7). For the field configuration (4) this winding number is given by the integer n. Note that the asymptotic condition (5*a*) agrees with (8).

That the solution (4) is localised to an absolute scale can easily be verified (for given f and V) by solving the linearised Euler–Lagrange equation for $\delta R = R - 1$, δR being a small deviation from the asymptotic value of R.

The solutions discussed so far are not minimal field configurations, in the sense that they do not satisfy the Bogomolniy equations saturating the inequalities (6*a*) and (6*b*). Indeed, the Bogomolniy equalities arising from (6*a*, *b*) are overdetermined and result in the trivial (vacuum) solution $R = 1$ only. There is, however, an interesting special case in which the Euler–Lagrange equations are solved by first-order Bogomolniy equations, which can be explicitly integrated.

This is the special case where the function f is equal to zero. In this case the Bogomolniy equation following from the inequality (6*a*) is

$$\lambda\sqrt{V} = \mathrm{i}\varepsilon_{ij}\partial_i\varphi\partial_j\varphi^* \tag{9}$$

which clearly solves the Euler–Lagrange equations pertaining to the $f = 0$ subsystem of (2). In this case the inequality (6*a*) is saturated, and for the field configuration (4), the Bogomolniy equation (9) reduces to the following integral:

$$n\int \frac{\mathrm{d}(R^2)}{\lambda\sqrt{V(R^2)}} = \tfrac{1}{2}r^2 + \text{constant} \tag{10}$$

which for suitable choices of the symmetry breaking potential $V(R^2)=V(|\varphi|,\eta)$, can be integrated explicitly, and, with a suitably normalised Lagrangian density, the corresponding action on $\mathbb{R}_2$ would be *equal* to 2π times the topological charge n, since (9) implies that this is a minimal field configuration. For example, with the simplest choice of $V=(\eta^2-|\varphi|^2)^2$, and setting the arbitrary constant on the right-hand side of (10) equal to zero, we have $R^2=1-\mathrm{e}^{-\lambda r^2/2}$, which satisfies the conditions (5*a*, *b*).

We complete our discussion for these classical lumps by commenting on the zero modes of Dirac equations in their background on $\mathbb{R}_2$.

It is well known [13] that the Dirac equation in the background of an Abelian-Higgs vortex on $\mathbb{R}_2$ can have zero modes only if the symbol of the Dirac equation features a term corresponding to the Yukawa interaction of the Dirac and Higgs fields. This is really because the symbol is descended from the four-dimensional Yang-Mills-Dirac system on $\mathbb{R}_2\times S^2$ by dimensional reduction [11]. (The same is true of the Dirac equation on $\mathbb{R}_3$.) Thus on $\mathbb{R}_2$ (as well as on $\mathbb{R}_3$) the symbol of the Dirac equation must feature both the gauge connection A_i and the Higgs field φ.

In our case for the system (2) on $\mathbb{R}_2$, the curious situation arises whereby we can find a non-trivial index (topological charge) with a Higgs field alone. In other words, the symbol of the Dirac equation in the background of our lump features only a Yukawa term with the Higgs field, and excludes the gauge connection term, and yet has normalisable zero modes. This is not surprising because the topological charge here belongs to a subsystem of the system descended from the dimensional reduction [11] of the generalised [12] Yang-Mills-Dirac system on $\mathbb{R}_2\times S^6$.

(II) *Quantum properties.* The quantum properties of our model given by the Lagrangian (1) are quite different from those of the $\mathbb{CP}^1$ sigma-model in (2+1) dimensions. While the latter system has a *local* U(1) gauge invariance, the system (1) has only a global U(1) gauge invariance, under $\varphi\to\mathrm{e}^{\mathrm{i}\alpha}\varphi$. This means that unlike the $\mathbb{CP}^1$ model, our model cannot be augmented by a Hopf term [10]. As such therefore, quantum mechanically our system is more akin to the Skyrme model in (3+1) dimensions rather than the $\mathbb{CP}^1$ model in (2+1) dimensions. (In some ways, our system is more akin to the Skyrme model also *classically*, namely that our lump in $\mathbb{R}_2$ is, like the Skyrmion in $\mathbb{R}_3$, localised to an absolute scale unlike the $\mathbb{CP}^1$ instanton.)

Based on this analogy, we adapt the discussion of the Skyrme model given by Adkins *et al* [14] to our model (1).

Restricting our attention to the rotational collective coordinate α, which happens to be the parameter of the global U(1) symmetry of (1), we promote α to a dynamical variable $\alpha(t)$ by allowing it to depend on time. Denoting our static lump configuration (4) by $\varphi_0=\varphi(r,\theta)$, we substitute $\varphi(r,\theta,t)=\varphi_0(r,\theta)\,\mathrm{e}^{\mathrm{i}\alpha(t)}$ into (1). The result is

$$L=-M+\tfrac{1}{2}\Lambda\dot{\alpha}^2 \tag{11}$$

where

$$M=2\pi\int r[\tfrac{1}{2}(\mathrm{i}\partial_{[i}\varphi_0\partial_{j]}\varphi_0^*)^2+f(\eta,|\varphi_0|)|\partial_i\varphi_0|^2+\lambda V(|\varphi_0|)]\,\mathrm{d}r \tag{12a}$$

$$\Lambda=4\pi\int r(\partial_i|\varphi_0|^2)^2\,\mathrm{d}r. \tag{12b}$$

Then, in terms of the canonical momenta $\pi=\partial L/\partial\dot{\alpha}=\Lambda\dot{\alpha}$, we have the Hamiltonian

$$H=M+\frac{1}{2\Lambda}\pi^2.$$

The canonical quantisation prescription $\pi=-\mathrm{i}\partial/\partial\dot{\alpha}$, allows diagonalisation of H, yielding the eigenfunctions $\psi(\alpha)=\exp \mathrm{i}a\alpha$, with eigenvalue a. Since $\alpha\rightarrow\alpha+2\pi\nu$ (ν integer) leaves $\psi(\alpha)$ unchanged, we would have bosonic quantisation if we chose also a to be an integer. This would yield eigenfunctions analogous to those of the z-component angular momentum. But here $\alpha(t)$ is not the azimuthal angle (θ in (4)) related to the Cartesian coordinates x, y. Therefore, in the absence of this 'physical' condition, we have no reason to constrain a to be an integer.

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